Quantum phase transitions in an array of coupled nanocavity quantum dots

Michal Grochol*
Department of Physics and Astronomy, Michigan State University, East Lansing, Michigan 48824, USA
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We investigate exciton-photon quantum phase transitions in a planar lattice of one-mode cavities containing one quantum dot. The quantum dot can be occupied by up to two excitons with opposite spin. We adopt the well-established mean-field approximation comprising an exciton order parameter and a photon coherence parameter. Calculating exciton- and photon-phase diagrams it is demonstrated that by controlling exciton- and photon-hopping energies a very rich scenario of coupled fermionic-bosonic quantum phase transitions, including, e.g., quantum phase transitions of the Hubbard model or quantum phase transitions of the light, is revealed. Moreover, our results support the interpretation of polariton Bose-Einstein condensation as only (polariton) laser coherence.

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I. INTRODUCTION

Since the dawn of quantum physics, there has been a long-standing interest in the quantum phase transition (QPT), i.e., a phase change by changing an external parameter at *zero temperature* driven by quantum fluctuations.[1] Recently, the QPT has attracted a lot of interest in many-body problems such as strongly interacting electronic systems in condensed-matter physics[2] or weakly interacting ultracold atomic systems.[3] Nevertheless, it is experimentally very demanding to observe such phenomena in these systems. It is somewhat easier in the case of optical lattices with ultracold atoms, allowing to conveniently study trapped Bose gases.[4]

Although photons are noninteracting Bose particles and they should always be found in the superfluid state,[5] the recent advances in the growth of cavities with high-quality factors have driven attention to novel light-matter system. The system of a two-dimensional array of coupled cavities with a single two-level atom inside each has been studied and shown to enable observation of QPTs of light, i.e., transition between Mott insulator (excitations localized on each site) and superfluid (excitations delocalized across the lattice) phase.[6–16] Such a system is usually described by the combination of Bose-Hubbard[5] or Jaynes-Cummings[17] model and solved by, e.g., mean-field approach,[7] density-matrix renormalization group,[18] or direct diagonalization.[12]

In this paper we investigate a system consisting of a two-dimensional array of one-mode polarization selective cavities containing one quantum dot (QD) as schematically shown in Fig. 1. There are either proposals for a system of coupled zero-dimensional cavities as mentioned above or for an array of quantum dots embedded in a planar microcavity.[19,20] In the latter case, the optical properties and the possible application of the structure for scalable quantum computation have been discussed in detail.[21] Furthermore, an array of quantum dots defined in a semiconductor two-dimensional electron-gas system has been proposed very recently as a voltage-controlled device for studying the fermionic Hubbard model[22] (originally introduced for the investigation of metal-insulator transitions[23]) with a long-range Coulomb interaction.[24] Our system can be realized as an array of QDs embedded in a photonic crystal, where the quantum dot is positioned on the site with a missing hole, i.e., zero-dimensional cavity—nanocavity.[7] Although the ex-

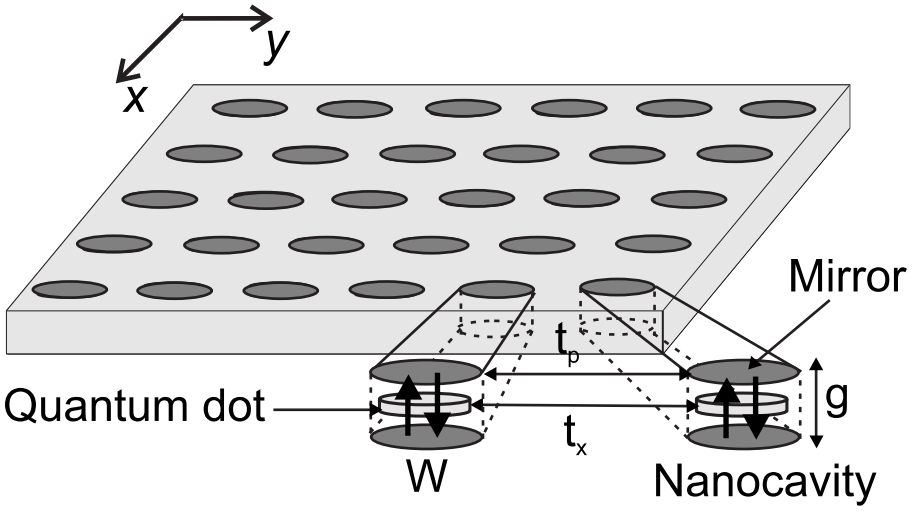

FIG. 1. Scheme of the investigated system: an array of the coupled nanocavity-quantum dots embedded in a photonic band-gap structure. Photon- (exciton-) hopping energy t_P (t_X), light-matter coupling g, and exciton-exciton interaction W are indicated.

perimental realization of the studied system might prove difficult in the near future, we believe that its theoretical investigation will shed light on two-component quantum phase transitions.

Furthermore, the investigated model can be also regarded as a tight-binding approach for the study of the polariton condensation in a system of the quantum well embedded in a planar cavity.[25] Our approach allows to assess the influence of different exciton- and photon-effective masses via the introduction (and the variation over several orders of magnitude) of the exciton hopping t_X and the photon-hopping t_P parameters. Thus, our model goes beyond the widely spread approach that always assumes photon coherence.[26] Consequently, it shows under which conditions the true polariton (generally two-component) condensation can occur. In the following, we nevertheless explicitly discuss only the nanocavity-quantum dot model but keep in mind that the quantum well polariton interpretation is also possible.

The Hamiltonian of the present system can be decomposed into the Hamiltonians of Hubbard and Jaynes-Cummings. However, the system can have both its components, fermionic (with exciton-spin projection depending on light polarization) and bosonic, simultaneously. Within the mean-field approximation for exciton order parameter and photon coherence parameter, more complicated phase diagrams (as functions of the exciton- and photon-hopping energies) with nontrivial interplay and correlation between different components (superfluid exciton or photon and insulating exciton or photon) arise. This goes clearly beyond the one-component model recently studied in Ref. 7. Nevertheless, in the limiting cases when one component is very weak, results very close to the well-known models of Hubbard or Bose-Hubbard are found.

This paper is organized as follows. In Sec. II we introduce the system and write down the Hamiltonian for the coupled nanocavity- (two-exciton) QD lattice together with some simplified calculations for zero exciton and photon hopping. The phase diagrams for various parameters are presented in Sec. III. Conclusions are drawn in Sec. IV. In the Appendix we provide the explicit form of the mean-field Hamiltonian.

II. THEORY

In order to concentrate on the most important (and from our point of view the most interesting) aspects of the system, we do not detail the precise material structure but rather write down our assumptions: (i) a strong exciton localization, which guarantees that maximally two excitons with opposite spins can be found in each quantum dot. The term exciton spin is used for the z (growth direction) component L_z of the total exciton angular momentum. There are four possibilities: two optically active (bright) excitons with $L_z \pm 1$ and two optically nonactive (dark) excitons with $L_z \pm 2$. In the following we restrict our discussion only to the bright ones. (ii) A spin-independent exciton energy ω_X, which for a small exciton-exciton distance is not guaranteed due to the exchange interaction and the vertex correction.[27,28] (iii) Existence of exciton transfer without specifying its dominant nature, i.e., Förster transfer[29] for short distances or radiative coupling[30,31] for long distances, and additionally we assume that the most important transfer is the one between the nearest neighbors. (iv) Existence of photon hopping among nanocavities (photonic band-gap structure), as recently discussed.[7,32] A schematic view of the proposed structure is shown in Fig. 1. The Hamiltonian in the second quantization with exciton $\hat{H}_X$, photon $\hat{H}_P$, and exciton-photon $\hat{H}_{XP}$ parts therefore reads ($\hbar=1$),

$$\hat{H}_X^{(1)} = \omega_X \sum_{j,\alpha} C_{j\alpha}^\dagger C_{j\alpha} - t_X \sum_{\langle j,k\rangle\alpha} C_{j,\alpha}^\dagger C_{k,\alpha} + W \sum_j C_{j\uparrow}^\dagger C_{j\uparrow} C_{j\downarrow}^\dagger C_{j\downarrow},$$

$$\hat{H}_P^{(1)} = \omega_C \sum_{j\alpha} a_{j\alpha}^\dagger a_{j\alpha} - t_P \sum_{\langle j,k\rangle\alpha} a_{j,\alpha}^\dagger a_{k,\alpha},$$

$$\hat{H}_{XP}^{(1)} = g \sum_{j\alpha} (a_{j\alpha}^\dagger C_{j\alpha} + a_{j\alpha} C_{j\alpha}^\dagger),$$

$$\hat{H} = \hat{H}_X^{(1)} + \hat{H}_P^{(1)} + \hat{H}_{XP}^{(1)} - \mu N, \tag{1}$$

where $C_{j\alpha}^\dagger$ is the exciton fermionic creation operator on the jth site with electron-spin projection $\alpha=\uparrow,\downarrow$, t_X (t_P) is the exciton (photon) intersite energy transfer between the nearest sites $\langle j,k\rangle$, $a_{j\alpha}^\dagger$ is the bosonic creation operator of the jth nanocavity photon mode with circular polarization α and energy ω_C, g is the light-matter coupling, W is the exciton-exciton interaction potential, μ is the chemical potential, and $N=\Sigma_{j\alpha}(C_{j\alpha}^\dagger C_{j\alpha}+a_{j\alpha}^\dagger a_{j\alpha})$ is the total number of particles.

The on-site nanocavity-laser coupling can be described in the quasimode approximation as $\hat{H}_j^{CL}=\Omega_\uparrow a_{j\uparrow}^\dagger+\Omega_\downarrow a_{j\downarrow}^\dagger$, with Ω_α being the coupling constant. Exciting the nanocavity with linearly polarized light $\Omega_\uparrow=\Omega_\uparrow$ and rotating the basis to $a_{jX}=\frac{1}{\sqrt{2}}(a_{j\uparrow}+a_{j\downarrow})$ and $a_{jY}=\frac{1}{\sqrt{2}}(a_{j\uparrow}-a_{j\downarrow})$ modifies the exciton-photon coupling to $\sim a_{jX}^\dagger(C_{j\uparrow}+C_{j\downarrow})$ and $\sim a_{jY}^\dagger(C_{j\uparrow}-C_{j\downarrow})$. Furthermore, we make a nontrivial assumption of a polarization selective cavity (although such a cavity has not been grown yet to the best of our knowledge). Consequently, assuming an X-polarization nanocavity all terms $\sim a_{jY}$ can be neglected since the exciton states couple either to X or Y polarization [X (Y) exciton $C_{X(Y)}^\dagger=\frac{1}{\sqrt{2}}(C_\uparrow^\dagger \pm C_\downarrow^\dagger)$]. Moreover, strong exciton spin-flip processes due to scattering are assumed to guarantee the same chemical potential for both spin-up and spin-down excitons. Finally, the following effective Hamiltonian is obtained:

$$\hat{H}_P^{(2)} = \omega_C \sum_j a_j^\dagger a_j - t_P \sum_{\langle j,k\rangle} a_j^\dagger a_k,$$

$$\hat{H}_{XP}^{(2)} = g \sum_{j\alpha} (a_j^\dagger C_{j\alpha} + a_j C_{j\alpha}^\dagger),$$

$$\hat{H} = \hat{H}_X^{(1)} + \hat{H}_P^{(2)} + \hat{H}_{XP}^{(2)} - \mu N, \tag{2}$$

where the operator of the number of particles changes to $N=\Sigma_{j\alpha}C_{j\alpha}^\dagger C_{j\alpha}+\Sigma_j a_j^\dagger a_j$. The sign and the magnitude of the exciton-exciton interaction potential W depend on the details

of electron and hole confinement in the QD.[33–35] The potential W can be tuned, e.g., by static electric field along the z direction or by applying a perpendicular magnetic field. Assuming that at zero magnetic field there is only a biexciton-binding energy $W=E_{XX}<0$, then for stronger magnetic field potential W changes by the effective exciton Zeeman splitting,[36] $W=E_{XX}+\mu_B g_X B_z$, where μ_B is the Bohr magneton, g_X is the exciton g factor, and B_z stands for the perpendicular magnetic field.

A close inspection of Hamiltonian Eq. (2) furthermore reveals that it comprises two well-known Hamiltonians: (i) the Hubbard Hamiltonian[22]

$$\hat{H}^H = \omega_X \sum_{j\alpha} C_{j\alpha}^\dagger C_{j\alpha} - t_X \sum_{\langle j,k\rangle\alpha} C_{j\alpha}^\dagger C_{k\alpha} + W \sum_j C_{j\uparrow}^\dagger C_{j\uparrow} C_{j\downarrow}^\dagger C_{j\downarrow}, \tag{3}$$

for which insulator-metal transitions have been intensively studied, and (ii) the Jaynes-Cummings Hamiltonian

$$\hat{H}_{j\alpha}^{JC} = \omega_X C_{j\alpha}^\dagger C_{j\alpha} + \omega_C a_j^\dagger a_j + g(a_j^\dagger C_{j\alpha} + a_j C_{j\alpha}^\dagger), \tag{4}$$

which has received a lot of attention recently.

In order to study the phase transition of model Eq. (2), we stay within the mean-field theory, which generally gives a very good description in accordance with the Monte Carlo simulation.[37] In analogy to classical phase transitions or Bogolyubov approach and the idea of (spontaneous) symmetry breaking, we introduce the *exciton superfluid order parameter* $\psi_\alpha=\langle C_{i\alpha}^\dagger\rangle$ and the *photon coherence parameter* $\chi=\langle a_j^\dagger\rangle$. Although, we note that the mean-field approach generally works better the higher the dimensionality of the system is, a two-dimensional system is still acceptable.[1] Thus, adopting the decoupling approximation

$$C_{i\alpha}^\dagger C_{j\alpha} = C_{i\alpha}^\dagger \langle C_{j\alpha}\rangle + \langle C_{i\alpha}^\dagger\rangle C_{j\alpha} - \langle C_{i\alpha}^\dagger\rangle\langle C_{j\alpha}\rangle,$$

$$a_i^\dagger a_j = a_i^\dagger \langle a_j\rangle + \langle a_i^\dagger\rangle a_j - \langle a_i^\dagger\rangle\langle a_j\rangle, \tag{5}$$

where i is not equal to j, Hamiltonian Eq. (2) in the mean-field approximation reads

$$\hat{H}_X^{(2)} = \sum_j \left(\omega_X \sum_\alpha C_{j\alpha}^\dagger C_{j\alpha} + W C_{j\uparrow}^\dagger C_{j\uparrow} C_{j\downarrow}^\dagger C_{j\downarrow} - z t_X \left[\sum_\alpha (C_{j\alpha}^\dagger \psi_\alpha + C_{j\alpha}\psi_\alpha^* - |\psi_\alpha|^2) \right] \right),$$

$$\hat{H}_P^{(3)} = \sum_j (\omega_C a_j^\dagger a_j - z t_P[(a_j^\dagger \chi + a_j \chi^*) - |\chi|^2]),$$

$$\hat{H}_{MF} = \hat{H}_X^{(2)} + \hat{H}_P^{(3)} + \hat{H}_{XP}^{(2)} - \mu N, \tag{6}$$

where z is the number of the nearest-neighbor dots. Another simplification is possible since we have performed test calculations for complex $\psi_\uparrow$, $\psi_\downarrow$, and χ, which have shown that (i) $\psi_\uparrow$ and $\psi_\downarrow$ are identical and (ii) *phase locking* between both parameters $\psi\equiv\psi_\uparrow=\psi_\downarrow$ and χ exists, which also follows from the symmetry of the Hamiltonian. This means that, e.g., in the case of energy it holds $E(\psi,\chi)=E(\psi e^{i\beta},\chi e^{i\beta})$, where β is an arbitrary real number. Recent studies of exciton phase transitions in coupled quantum wells have shown that depending on the sign of the exciton-exciton interaction W, the exciton ground state can be either paramagnetic ($\psi_\uparrow=\psi_\downarrow$ for $W<0$) or ferromagnetic ($\psi_\uparrow\neq 0$ and $\psi_\uparrow=0$ for $W>0$).[33,34] However, in the present case the light-matter coupling g plays a very important role tending to equalize spin populations if it is sufficiently strong ($g\sim W$). It implies the convenient usage of a spin-independent order parameter $\psi_\uparrow=\psi_\downarrow=\psi$. Furthermore, we note that the reality of ψ and χ is a well-established property of the mean-field[38] if they are independent. In the present case, they are coupled only indirectly via the light-matter coupling $\sim g$, which additionally explains the phase locking. Put all, the mean-field Hamiltonian Eq. (6) is modified as

$$\hat{H}_X^{(3)} = \sum_j \left(\omega_X \sum_\alpha C_{j\alpha}^\dagger C_{j\alpha} + W C_{j\uparrow}^\dagger C_{j\uparrow} C_{j\downarrow}^\dagger C_{j\downarrow} - z t_X \left[\sum_\alpha (C_{j\alpha}^\dagger + C_{j\alpha})\psi - 2|\psi|^2) \right] \right),$$

$$\hat{H}_P^{(4)} = \sum_j (\omega_C a_j^\dagger a_j - z t_P[(a_j^\dagger + a_j)\chi - |\chi|^2]),$$

$$\hat{H}_{MF} = \hat{H}_X^{(3)} + \hat{H}_P^{(4)} + \hat{H}_{XP}^{(2)} - \mu N. \tag{7}$$

We note that if $\chi\neq 0$ photons are in a coherent state and if $\chi=0$ they are in a Fock state. For convenience, the exciton-photon *detuning* can be defined as

$$\Delta = \omega_X - \omega_C. \tag{8}$$

The case without exciton and photon transfers ($t_X=0$ and $t_P=0$), i.e., exciton and photon insulators, can be solved easily. One only needs to diagonalize the Hamiltonian matrix, which in the basis of n excitations $|1,0,n-1\rangle$, $|0,1,n-1\rangle$, $|1,1,n-2\rangle$, and $|0,n\rangle$ with the notation $|n_{X\uparrow},n_{X\downarrow},n_P\rangle$, where $n_{X\alpha}$ (n_P) is the exciton (photon) number, takes the form of

$$H_{IN} = \begin{pmatrix} \Delta & 0 & \sqrt{n-1}g & \sqrt{n}g \\ 0 & \Delta & \sqrt{n-1}g & \sqrt{n}g \\ \sqrt{n-1}g & \sqrt{n-1}g & 2\Delta+W & 0 \\ \sqrt{n}g & \sqrt{n}g & 0 & 0 \end{pmatrix}, \tag{9}$$

with the ground-state energy $E_{IN}(n,\Delta)$. Its knowledge, as a function of the number of particles or detuning, enables to calculate the chemical potential from its definition as an energy needed to add a new particle into the system

$$\mu(n,\Delta) = E_{IN}(n+1,\Delta) - E_{IN}(n,\Delta). \tag{10}$$

This is the starting point of our analysis of phase diagrams in Sec. III and it is shown in Fig. 2.

Before we focus on the numerical results, we note that in order to explore experimentally the phase diagrams shown in Sec. III, it is necessary that exciton- and photon-transfer energies t_X and t_P vary over several orders of magnitude. In the exciton case, as already mentioned above, this can be achieved, e.g., by modifying the Förster transfer[29] or by applying external fields.[39] Varying photon transfer energies is however more difficult since assuming that photon hopping limits the cavity quality factor Q then it holds $t_P=\omega_C/Q$.[32]

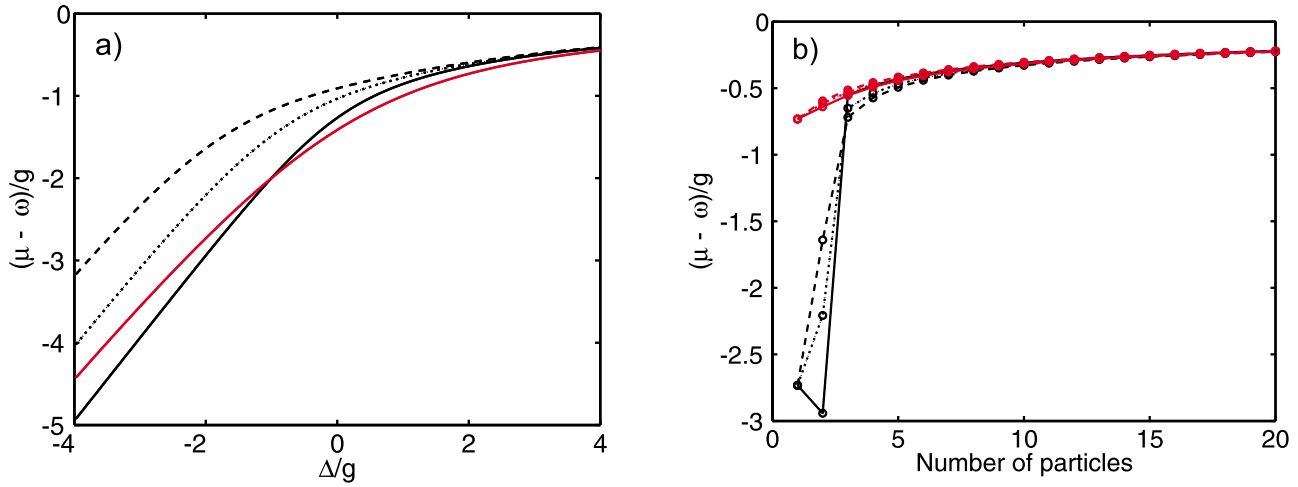

FIG. 2. (Color online) (a) Chemical potential as a function of exciton-photon detuning Δ for $t_X=0$ and $t_P=0$ and for different number of particles $n=1$ (red/gray) and $n=2$ (black). (b) Chemical potential as a function of the number of particles for different detunings $\Delta=-2g$ (black) and $\Delta=2g$ (red/gray) and for $t_X=0$ and $t_P=0$. Various on-site energies $W=-g$ (solid), $W=0$ (dotted), and $W=g$ (dashed) are taken into account.

Consequently, a post-creation precise control of the quality factor of each cavity is required.

III. RESULTS AND DISCUSSION

Here we present numerically calculated phase diagrams (order and coherence parameters) and numbers of excitons or photons as functions of chemical potential, and exciton- and photon-hopping energy, t_X and t_P, respectively. We have taken into account up to 20 particles (either excitons or photons) and minimized the system energy with respect to the order parameter ψ and the coherence parameter χ.

Let us start looking at the dependence of chemical potential μ on detuning Δ and number of particles n, which is shown in Fig. 2. We can notice that for negative detunings and biexciton case $W<0$, it holds for the chemical potential, $\mu(2,\Delta)<\mu(1,\Delta)$, and the system changes its state directly from $n=0$ to $n=2$ without passing through one-particle state. In the case of repulsion (no biexciton), there are distinguishable transitions from zero to one and from one to two particles, which, however, get closer with positive Δ as the particles tend to be more photonlike and therefore, less sensitive to the exciton potential W. The transitions for more particles $n>2$ are smooth as only the number of photons is increased (having always two excitons). In the limiting case of a very large number of particles $n\to\infty$, the chemical potential approaches zero $\mu(\infty,\Delta)\to 0$ and consequently the system approaches the noninteracting (photon) Bose gas.

Before we start to discuss the mean-field results, we note that they are valid only for $\mu<\omega_C$ since for $\mu\to\omega_C$ the method does not converge. However, in the case of $\mu\to\omega_C$ the system is superfluid. We further note that the exciton spontaneous emission rate γ_X or the photon-leakage rate out of the nanocavity κ lead to an effective broadening of eigenstates. In order to observe Mott lobes, the following condition has to be fulfilled: $\mu(n+1,\Delta)-\mu(n,\Delta)>\Gamma$, with Γ being the ground-state broadening. This means that effectively only the lowest Mott lobes are easy to observe.

A. Exciton phase diagrams

We begin with exciton phase diagrams. We note that without light-matter interaction, $g=0$, and with exciton-exciton repulsion, $W>0$, the mean-field results of the Hubbard model,[22] i.e., three Mott lobes instead of two (as we will see), are retrieved.

First, we take a very small photon hopping $t_P \ll g$ and focus on phase diagrams for the order parameter ψ for negative detuning Δ as a function of chemical potential μ and normalized exciton-transfer energy t_X/g, which are depicted in Fig. 3(a). Due to our choice of photon hopping the system is effectively only excitonic for most values of the chemical potential. There are clearly visible Mott lobes with $\psi=0$ for small t_X and superfluid phase with $\psi\neq 0$ for large t_X. The boundaries of Mott lobes for a small transfer t_X can be identified from the dependence of the chemical potential on the number of particles in Fig. 2. In the biexciton case $W<0$, there are only two lobes, either without any particle or with a biexciton as seen in Fig. 3(a). In the case of exciton repulsion, there would be three lobes with zero, one, and two particles, respectively. This would resemble either the results of the pure Hubbard model[5] or the case of coupled cavities although in that case there are many modes since there are no limitations on the number of photons.[7] Furthermore, the number of photons starts to increase only as the chemical potential approaches zero in correspondence with the results seen in Fig. 2. Moreover, for these values of chemical potential the photon superfluidity appears. This means that order and coherence parameters are *correlated* in a nontrivial way. In the one-component system, only the absolute value of the order or coherence parameter is fixed and usually the sign is chosen due to the intuitive physical interpretation $\sqrt{n_{j\alpha}}=\langle C_{j\alpha}\rangle=\langle C_{j\alpha}^{\dagger}\rangle$. Here, this unambiguity is partially broken

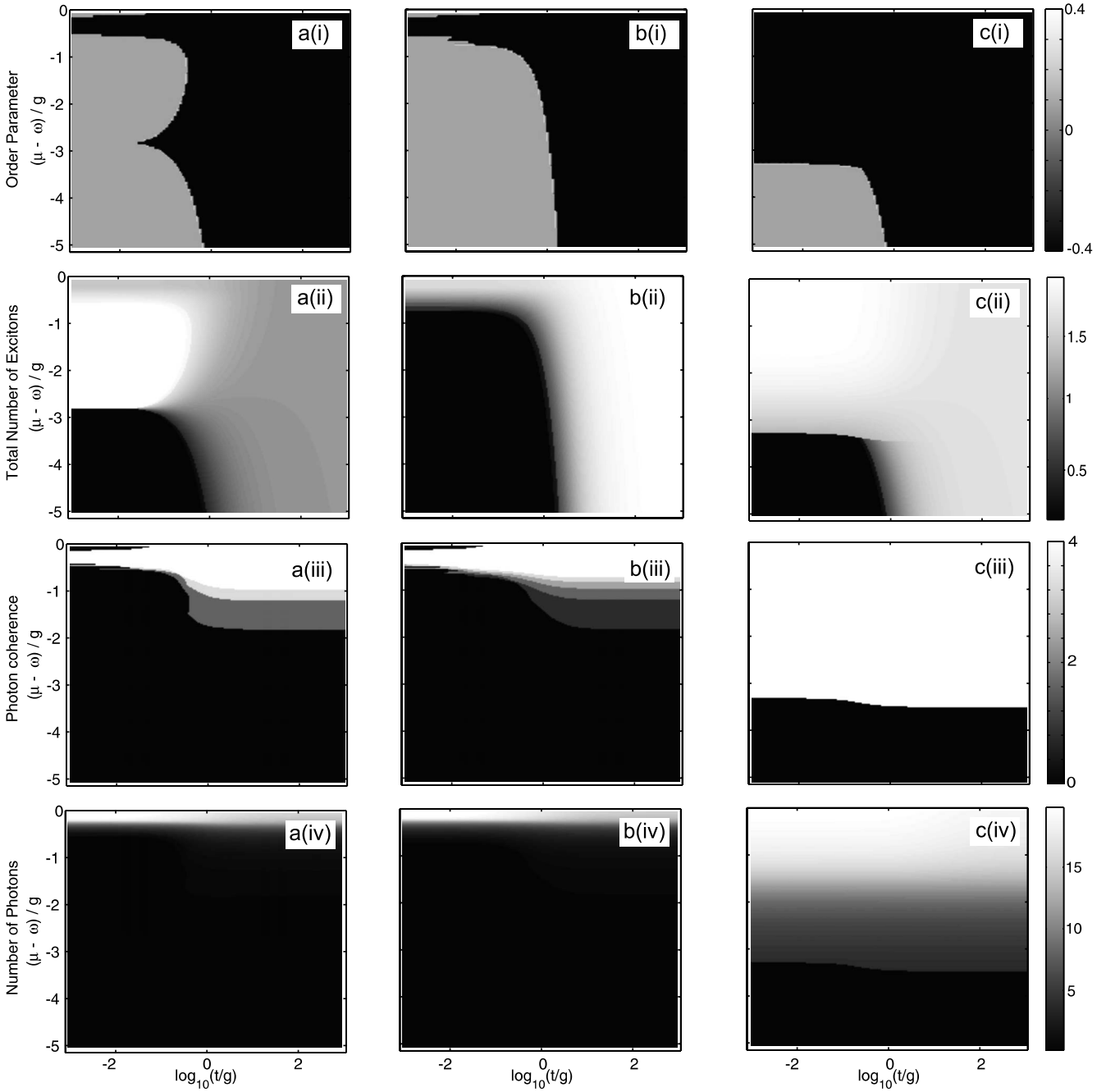

FIG. 3. (i) Order parameter ψ, (ii) number of excitons n_X, (iii) photon coherence parameter χ, and (iv) number of photons n_P as functions of the exciton tunneling t_X and chemical potential μ for negative detuning [(a) and (c)] $\Delta=-2g$ and (b) $\Delta=2g$ for on-site exciton energy $W=-g$ and for photon hoppings [(a) and (b)] $t_P=10^{-3}g$ and (c) $t_P=g$. In the case of a nonconstant behavior, the insulating (superfluid) phases, i.e., $\psi=0$ ($\psi\neq 0$) or $\chi=0$ ($\chi\neq 0$), are found on the left-hand (right-hand) side.

since only one of the parameters can be positive, i.e., a new additional condition for energy minimum appears

$$\psi\chi < 0. \tag{11}$$

One can thus choose $\psi>0$, which implies $\chi<0$ and the naive interpretation of the coherence parameter as $\chi=\sqrt{n_j^P}$, n_j^P being the number of photons, is lost.

Additionally, the case of positive detuning is shown in Fig. 3(b). In this case results for different exciton potentials W are similar since there are only two exciton phases: (i) insulating without any particle and (ii) superfluid. As in the previous case, depending on whether there are superfluid photons or not in the system, the signs of the order and coherence parameters are correlated. Moreover, the number of particles increases with μ or when the total exciton energy becomes comparable to the lower photon energy due to increased transfer t_X.

Second, we proceed further by increasing the photon hopping to the level of light-matter coupling $t_P=g$ (medium

value of hopping). Corresponding results are plotted in Fig. 3(c). Unlike in the first case, profound differences to the scenario which are usually seen for fermionic or bosonic Hubbard models are found. The fact that the system consists of two kinds of particles, fermionic excitons and bosonic photons, manifests itself strongly since there are clearly two ways of changing the ground-state nature: (i) insulator-superfluid transition as the exciton order parameter changes from $\psi=0$ to $\psi\neq 0$ with increasing exciton hopping t_X; (ii) transition from the purely excitonic ground state (caused by negative detuning) with $\psi=0$ or $\psi\neq 0$ for small chemical potential μ to the mixed exciton-photon state with sign correlation for a larger chemical potential μ. Moreover, photons (if present) are always found in the superfluid phase since the value of photon hopping, $t_P=g$, is sufficient to overcome the on-site effective photon repulsion due to the coupling to excitons.[7] The increase in photon number slightly depends on exciton hopping t_X—the larger it is the more preferable excitons are.

Increasing the photon hopping even further, $t_P \gg g$, leads the system to the purely photonic and superfluid phase as expected (not shown). This case is interesting from the point of view of the polariton (exciton-photon) Bose-Einstein condensation (BEC) in microcavities whose investigation has gained a considerable interest in recent years both theoretically (see Ref. 40 and references therein) and experimentally.[41–45] Nevertheless, in the theoretical investigation of polariton condensation[46–48] it is always assumed that the photon field is superfluid and the exciton part is further investigated. Even though the assumption of photon superfluidity is well justified in planar microcavities, where $t_P \gg t_X$ (Ref. 25) (as in the current case), it leads to the conclusion that polariton condensation is more similar (possibly identical) to polariton laser coherence,[49,50] i.e., $\chi\neq 0$, than to (nonequilibrium) Bose-Einstein condensate of the matter.[51] In other words, if photon coherence (superfluidity) disappeared there would be no condensation. This critical point of view has been recently formulated by Butov in Ref. 52. As our results show the exciton superfluidity would remain for a sufficiently small exciton mass, approximately holding $g \sim t_X \sim \frac{\hbar^2}{2m_X d^2}$, where d is the interdot distance and m_X is the effective exciton mass. For reasonable values of $g=0.1$ meV and $d=100$ nm, one obtains that a very light exciton mass of $m_X=0.03m_0$ (m_0 being the bare-electron mass) is needed. Such a light mass is difficult to find in typical semiconductor nanostructures.

We may also ask if condensation of any composite system (e.g., exciton photon) is possible at all. Our results show that genuine condensation, where coherence is present for both particles simultaneously, is possible only if the hopping energy of both constituents does not differ very much (within one order of magnitude). From this perspective, it is also understandable that one can observe exciton Bose-Einstein condensation since, in the language of the present model, hopping energies of electron and hole are within the same order of magnitude. We note that the investigation of exciton BEC in the coupled quantum wells has gained a lot of experimental interest recently.[53,54]

B. Photon phase diagrams

After studying phase transitions from the exciton perspective we now turn our attention toward the photon perspective. As already mentioned in Sec. I quantum phase transitions of light have become a very active field of research only very recently. Although we are going to plot the results in the same way as for QPT of light it is important to keep in mind that we are dealing with a coupled exciton-photon system.

As in Sec. III A we begin with the almost one-particle system, i.e., $t_X \ll g$, with the results depicted in Fig. 4(a). Indeed, these results resemble very much those for the system studied for quantum phase transitions of light. Especially, the plot of photon coherence in Fig. 4(a) shows many common features, with Fig. 4c in Ref. 7 also calculated for negative detuning. A clear insulator-superfluid transition for photons can be observed. However, there are two main differences to previous studies: (i) in the absence of photons, the system behaves as an exciton system, i.e., due to our choice of the exciton-exciton interaction, as mentioned above, there is an immediate transition from zero to two excitons (biexciton) with increasing chemical potential μ. (ii) As we have seen before, the order and the coherence parameters are correlated and thus, with superfluid photons found in the system, the exciton is also in the superfluid phase. Nevertheless, with increasing photon hopping t_P the exciton component disappears.

Furthermore, results for an increased exciton-hopping energy, $t_X=g$, are shown in Fig. 4(b). An insulator-superfluid transition of light can again be nicely seen, which is only very slightly modified by the increased exciton hopping with respect to the results plotted in Fig. 4(a). The increased exciton hopping t_X leads to the fact that for small photon hopping t_P there is only superfluid exciton found in the system ($\psi\neq 0$) and as soon as the superfluid photon appears the sign correlation is established. The exciton then gradually disappears as the photon hopping t_P, and consequently the photon number n_P, is increased.

Last but not least, the exciton-hopping energy is increased even further and results are shown in Fig. 4(c). The scenario from the previous case is repeated. However, the superfluid exciton is found in the system for all values of the hopping t_P since a sufficiently strong value, which is needed to dominate and force the system to be purely photonic and superfluid, is not reached. Finally, we mention that the (energy, oscillator strength, and positional) disorder[20] would introduce a third phase similar to the Bose glass in the case of the Bose-Hubbard model.[5]

IV. CONCLUSIONS

In summary, we have investigated exciton-photon quantum phase transitions. The studied system consists of a planar lattice of one-mode nanocavity (two-exciton) quantum dots. Staying in the mean-field approximation and introducing exciton order parameter and photon coherence parameter, we have investigated a variety of parameter combinations and shown the cases of (i) an almost pure exciton (Hubbard-type) quantum phase transition, (ii) an almost pure photon

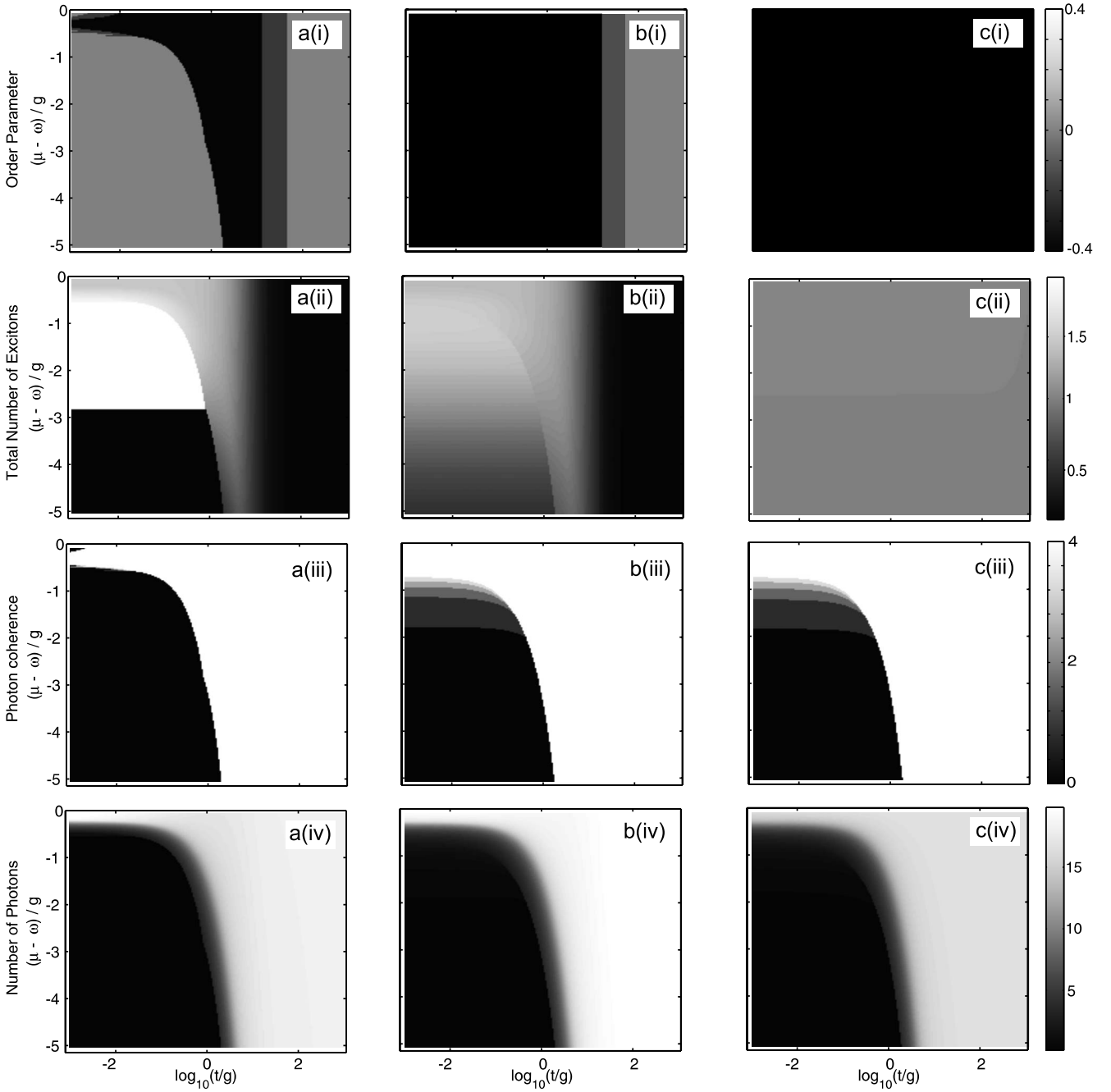

FIG. 4. (i) Order parameter ψ, (ii) number of excitons n_X, (iii) photon coherence parameter χ, and (iv) number of photons n_P as functions of photon tunneling t_P and chemical potential μ for negative detuning $\Delta=-2g$ for on-site exciton energy $W=-g$ and for exciton hoppings (a) $t_X=10^{-3}g$, (b) $t_X=g$, and (c) $t_X \gg g$. In the case of a nonconstant behavior, the insulating (superfluid) phases, i.e., $\psi=0$ ($\psi\neq 0$) or $\chi=0$ ($\chi \neq 0$), are found on the left-hand (right-hand) side.

(Bose-Hubbard-type) quantum phase transition, and (iii) the complicated case of a mixed exciton-photon transition where light-matter coupling is comparable to exciton or photon hopping. It has been clearly demonstrated that exciton order parameter and photon coherence parameter are correlated leading to additional phase transitions. We have discussed our results in relation to the ongoing debate about the nature of polariton condensation and argued that they support the idea of a polariton laser.

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APPENDIX: HAMILTONIAN MATRIX

For completeness, Hamiltonian Eq. (7) can also be written

in the matrix form. After rotating the exciton basis into the bright and dark ones $[C_{jB(D)}=\frac{1}{\sqrt{2}}(C_{j\uparrow}\pm C_{j\downarrow})]$, basis vectors are chosen as $\{|00\rangle,|10\rangle,|01\rangle,|20\rangle,|11\rangle,|02\rangle,|21\rangle,|12\rangle,|03\rangle\cdots\}$ with the notation $|n_X,n_P\rangle$, where n_X (n_P) is the bright exciton (photon) number. Furthermore, renormalizing the exciton order parameter $\tilde{\psi}=\sqrt{2}\psi$ and the light-matter coupling $\tilde{g}=\sqrt{2}g$ and setting the energy zero to the value of ω_C, the Hamiltonian matrix takes the form of

$$H_{MF}^{(2)}=\begin{pmatrix}
0 & -zt_X\tilde{\psi} & -zt_P\chi & 0 & 0 & 0 & 0 & 0 & 0 \\
-zt_X\tilde{\psi} & \Delta-\mu & \tilde{g} & -zt_X\tilde{\psi} & -zt_P\chi & 0 & 0 & 0 & 0 \\
-zt_P\chi & \tilde{g} & -\mu & 0 & -zt_X\tilde{\psi} & -zt_P\chi & 0 & 0 & 0 \\
0 & -zt_X\tilde{\psi} & 0 & 2\Delta+W-2\mu & \tilde{g} & 0 & -zt_P\chi & 0 & 0 \\
0 & -zt_P\chi & -zt_X\tilde{\psi} & \tilde{g} & \Delta-2\mu & \sqrt{2}\tilde{g} & -zt_X\tilde{\psi} & -zt_P\chi & 0 \\
0 & 0 & -zt_P\chi & 0 & \sqrt{2}\tilde{g} & -2\mu & 0 & -zt_X\tilde{\psi} & -zt_P\chi \\
0 & 0 & 0 & -zt_P\chi & -zt_X\tilde{\psi} & 0 & 2\Delta+W-3\mu & \sqrt{2}\tilde{g} & 0 \\
0 & 0 & 0 & 0 & -zt_P\chi & -zt_X\tilde{\psi} & \sqrt{2}\tilde{g} & \Delta-3\mu & \sqrt{3}\tilde{g} \\
0 & 0 & 0 & 0 & 0 & -zt_P\chi & 0 & \sqrt{3}\tilde{g} & -3\mu \\
 & & & & & & & & \ddots
\end{pmatrix}+zt_X|\tilde{\psi}|^2I+zt_P|\chi|^2I, \tag{A1}$$

where **I** is the identity matrix.

*Present address: Institut für Theoretische Physik Interdiziplinaeres-Zentrum für Molekulare Materialien (ICMM) Friedrich-Alexander-Universität Erlangen-Nürnberg Staudstr. 7, D-91058 Erlangen, Germany.

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